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$A_n^{(1)}$ affine Toda field theories with integrable boundary conditions revisited

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Abstract: Generic classically integrable boundary conditions for the $A_n^{(1)}$ affine Toda field theories (ATFT) are investigated. The present analysis rests primarily on the underlying algebra, defined by the classical version of the reflection equation. We use as a prototype example the first non-trivial model of the hierarchy i.e. the $A_2^{(1)}$ ATFT, however our results may be generalized for any $A_n^{(1)}$ $(n > 1)$. We assume here two distinct types of boundary conditions called some times soliton preserving (SP), and soliton non-preserving (SNP) associated to two distinct algebras, i.e. the reflection algebra and the (q) twisted Yangian respectively. The boundary local integrals of motion are then systematically extracted from the asymptotic expansion of the associated transfer matrix. In the case of SNP boundary conditions we recover previously known results. The other type of boundary conditions (SP), associated to the reflection algebra, are novel in this context and lead to a different set of conserved quantities that depend on free boundary parameters. It also turns out that the number of local integrals of motion for SP boundary conditions is 'double' compared to those of the SNP case.

Keywords: Integrable Hierarchies, Integrable Field Theories.

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Contents

1. Introduction

Integrability in the bulk has admittedly attracted a great deal of research interest in recent years, however after the seminal works of [1 – 3] particular emphasis has been given on the issue of incorporating consistent boundary conditions in integrable models. This shed new light into the bulk theories themselves, and also opened the path to new mathematical concepts and physical applications. In a more general setting the investigation of both classical and quantum integrable systems, particularly those with non-trivial boundary conditions, turns out to be quite significant especially after the recent advances within the AdS/CFT correspondence [4] uncovering the important role of integrability [5]. A crucial question within this frame is what would the physical implications be in both gauge and string theories once non-trivial consistent boundary conditions, especially the ones that may modify the bulk behavior, are imposed to the associated lattice and continuum integrable models (for some recent results see [6] and references therein). Therefore studies concerning the existence of consistent boundary conditions that preserve integrability are of particular significance and timeliness not only for the integrable systems themselves, but for other active research fields.

The central purpose of the present article is the investigation of classical integrable models when general boundaries that preserve integrability are implemented. Among the various classes of integrable models we choose to consider here a particular class that is the affine Toda field theories (ATFT) [7, 8]. The prototype model of this class is the

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sine-Gordon model, which has been extensively studied both in the bulk [9] as well as in the presence of non-trivial integrable boundary conditions [10]. Generic affine Toda field theories with classical integrable boundary conditions were first analyzed more than a decade ago in [11]. A different point of view, although regarding the same class of boundary conditions[1] analyzed in [11], is presented in [12]. Specifically, in [12] the $A_2^{(1)}$ ATFT with 'dynamical' boundary conditions –that is a quantum mechanical system is attached at the boundary— is investigated. Further studies regarding the boundary ATFT at both classical and quantum level may be also found in various articles (see e.g. [13]–[18]).

Although the analysis in [11] seems quite exhaustive it turns out that in simply-laced ATFT a whole class of consistent boundary conditions is absent. Our main objective here is to systematically search for all possible boundary conditions in $A_n^{(1)}$ ATFT and eventually implement the missing ones. More precisely, we assume two distinct types of boundary conditions called soliton preserving (SP), and soliton non-preserving (SNP) associated to two distinct algebras, i.e. the reflection algebra [2] and the twisted Yangian [19, 20] respectively (see also relevant studies in [18, 21 – 25]).

Depending on the choice of boundary conditions certain physical behavior is entailed. Specifically, in the context of imaginary $A_n^{(1)}$ ATFT the boundary conditions introduced in [11], known as SNP, oblige a soliton to reflect to an anti-soliton. In real $A_n^{(1)}$ ATFT on the other hand such boundary conditions lead to the reflection of a fundamental particle to itself. Recall that fundamental particles in real ATFT are equivalent to the lightest bound states (breathers) of the imaginary theory provided that $\beta \to i\beta$ (β is the coupling constant of the theory). It is however clear that another possibility arises, that is the implementation of certain boundary conditions that lead to the reflection of a soliton to itself in imaginary ATFT or to the reflection of a fundamental particle to its conjugate in real ATFT. These boundary conditions are known as soliton preserving and have been extensively analyzed in the frame of integrable quantum spin chains [21, 22, 26]–[30].

Notwithstanding SP boundary conditions are somehow the obvious ones in the framework of integrable lattice models they have remained elusive in the context of $A_n^{(1)}$ ATFT for quite a long time. Note however that in quantum spin chains in addition to the well studied SP boundaries SNP boundary conditions were first introduced in [31] and further analyzed and generalized in [21, 22, 24, 25]. It is thus our primary objective here to complete the study of integrable boundary conditions in ATFT by introducing and fully analyzing the novel (SP) boundary conditions.

The outline of this article is as follows: in the next section we present the basic preliminary notions regarding the algebraic setting for classical models on the full line and on the interval. In our analysis we adopt the line of attack described in e.g. [32], and in [33, 34] for boundary systems. More precisely we introduce the classical Yang-Baxter equation and the underlying algebra for the system on the full line. In the situation of a system on the interval we distinguish two types of boundary conditions based on the classical versions of the reflection algebra (SP) and (q) twisted Yangian (SNP). Next the

[1] by 'same class of boundary conditions' we mean that in both studies [11, 12] a common underlying algebra –(classical) q-twisted Yangian– is implicitly assumed. Note however that the analysis in [11] is classical while in [12] is quantum.

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$A_n^{(1)}$ ATFT on the full line is reviewed and an explicit derivation of the local integrals of motion by solving the auxiliary linear problem [32] is presented. In section 3 being guided by the same logic and adopting Sklyanin's formulation [2] we rederive the integrals of motion of the $A_n^{(1)}$ ATFT with SNP boundary conditions. Note that analogous strategy was followed in [33] and [34] for the classical boundary sine-Gordon and vector NLS models respectively. Our results are in agreement with the ones deduced in [11]. In section 4 we introduce for the first time the novel boundary conditions (SP) within the context of ATFT. Explicit expressions of the associated local integrals of motion are deduced from the asymptotic expansion of the classical transfer matrix. It is worth stressing that the induced integrals of motion depend on free boundary parameters as opposed to the SNP case. In the last section a discussion on the entailed results is presented and several directions for future investigations are proposed.

2. Preliminaries

The analysis of the ATFT with integrable boundary conditions will rely on the solution of the so called auxiliary linear problem [32]. Before we proceed to the study of classical integrable models with consistent boundary conditions it will be instructive to recall the basic notions in the periodic case. Let Ψ be a solution of the following set of equations

$$\frac{\partial \Psi}{\partial x} = \mathbb{U}(x,t,\lambda)\Psi \tag{2.1}$$

$$\frac{\partial \Psi}{\partial t} = \mathbb{V}(x,t,\lambda)\Psi \tag{2.2}$$

with $\mathbb{U},\ \mathbb{V}$ being in general $n \times n$ matrices with entries functions of complex valued fields, their derivatives, and the spectral parameter λ. Compatibility conditions of the two differential equation (2.1), (2.2) lead to the zero curvature condition [35 – 37]

$$\dot{\mathbb{U}} - \mathbb{V}' + \Big[\mathbb{U},\ \mathbb{V}\Big] = 0. \tag{2.3}$$

The latter equations give rise to the corresponding classical equations of motion of the system under consideration. The monodromy matrix from (2.1) may be written as:

$$T(x,y,\lambda) = \mathcal{P}exp\left\{\int_y^x \mathbb{U}(x',t,\lambda)dx'\right\} \tag{2.4}$$

with $T(x,x,\lambda) = 1$. The monodromy matrix satisfies apparently (2.1), and this will be extensively used in the present analysis. On the other hand within the Hamiltonian formalism the existence of the classical r-matrix, satisfying the classical Yang-Baxter equation [38, 39]

$$[r_{12}(\lambda_1 - \lambda_2),\ r_{13}(\lambda_1) + r_{23}(\lambda_2)] + [r_{13}(\lambda_1),\ r_{23}(\lambda_2)] = 0, \tag{2.5}$$

guarantees the integrability of the classical system. Indeed, consider the operator $T(x,y,\lambda)$ satisfying

$$\{T_1(x,y,t,\lambda_1),\ T_2(x,y,t,\lambda_2)\} = [r_{12}(\lambda_1 - \lambda_2),\ T_1(x,y,t,\lambda_1)T_2(x,y,t,\lambda_2)]\,. \tag{2.6}$$

Making use of the latter equation one may readily show for a system in full line:

$$\{\ln tr\{T(x,y,\lambda_1)\},\ \ln tr\{T(x,y,\lambda_2)\}\} = 0 \tag{2.7}$$

i.e. the system is integrable, and the charges in involution –local integrals of motion– may be obtained by expanding the object $\ln tr\{T(x,y,\lambda)\}$.

The classical r-matrix associated to the $A_n^{(1)}$ affine Toda field theory in particular is given by[2] [40]

$$r(\lambda) = \frac{\cosh(\lambda)}{\sinh(\lambda)} \sum_{i=1}^{n+1} e_{ii} \otimes e_{ii} + \frac{1}{\sinh(\lambda)} \sum_{i\neq j=1}^{n+1} e^{[sgn(i-j)-(i-j)\frac{2}{n+1}]\lambda} e_{ij} \otimes e_{ji}. \tag{2.8}$$

Note that the classical r-matrix (2.8) is written in the so called principal gradation as is also in [11, 17]. To express the r-matrix in the homogeneous gradation one implements a simple gauge transformation:

$$r^{(h)}(\lambda) = \mathcal{V}(\lambda)\ r^{(p)}(\lambda)\ \mathcal{V}(-\lambda) \quad \text{where} \quad \mathcal{V}(\lambda) = \sum_{j=1}^{n+1} e^{\frac{2(j-1)\lambda}{n+1}} e_{jj}. \tag{2.9}$$

Our main aim as mentioned upon is to study the $A_n^{(1)}$ ATFT on the interval. For this purpose we shall employ Sklyanin's formulation (see also [33, 34] for classical models with integrable boundary conditions). It will be convenient for our purposes here to introduce some useful notation:

$$\begin{array}{llll}
\hat{r}_{ab}(\lambda) = r_{ba}(\lambda) & \text{for SP,} & \hat{r}_{ab}(\lambda) = r_{ba}^{t_a t_b}(\lambda) & \text{for SNP} \\
r^*_{ab}(\lambda) = r_{ab}(\lambda) & \text{for SP,} & r^*_{ab}(\lambda) = r_{ba}^{t_b}(-\lambda) & \text{for SNP} \\
\hat{r}^*_{ab}(\lambda) = r_{ba}(\lambda) & \text{for SP,} & \hat{r}^*_{ab}(\lambda) = r_{ab}^{t_a}(-\lambda) & \text{for SNP} \\
T(\lambda) = T^{-1}(-\lambda) & \text{for SP,} & \hat{T}(\lambda) = T^t(-\lambda) & \text{for SNP.}
\end{array} \tag{2.10}$$

In the situation where non-trivial integrable boundary conditions are implemented one derives two types of 'monodromy' matrices $\mathcal{T}$, which respectively represent the classical versions of the reflection algebra $\mathbb{R}$, and the twisted Yangian $\mathbb{T}$ written in the compact form below (see e.g. [2, 41]):

$$\begin{aligned}
\Big\{\mathcal{T}_1(\lambda_1),\ \mathcal{T}_2(\lambda_2)\Big\} = {} & r_{12}(\lambda_1-\lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2) - \mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)\hat{r}_{12}(\lambda_1-\lambda_2) \\
& + \mathcal{T}_1(\lambda_1)\hat{r}^*_{12}(\lambda_1+\lambda_2)\mathcal{T}_2(\lambda_2) - \mathcal{T}_2(\lambda_2)r^*_{12}(\lambda_1+\lambda_2)\mathcal{T}_1(\lambda_1).
\end{aligned} \tag{2.11}$$

The modified 'monodromy' matrices, compatible with the corresponding algebras $\mathbb{R}$, $\mathbb{T}$ are given by the following expressions [2, 11]:

$$\mathcal{T}(x,y,t,\lambda) = T(x,y,t,\lambda)\ K^-(\lambda)\ \hat{T}(x,y,t,\lambda) \tag{2.12}$$

and the generating function of the involutive quantities is defined as

$$t(x,y,t,\lambda) = tr\{K^+(\lambda)\ \mathcal{T}(x,y,t,\lambda)\} \tag{2.13}$$

[2]Notice that the r-matrix employed here is in fact $r_{12}^{t_1 t_2}$ with r_{12} being the matrix used e.g. in [28, 23]

where $K^{\pm}$ c-number representations of the algebra $\mathbb{R}$ ($\mathbb{T}$) satisfying (2.11) for SP and SNP respectively, and also

$$\left\{K_1^{\pm}(\lambda_1),\ K_2^{\pm}(\lambda_2)\right\} = 0. \tag{2.14}$$

Due to (2.11) it can be shown that

$$\{t(x,y,t,\lambda_1),\ t(x,y,t,\lambda_2)\} = 0, \quad \lambda_1,\ \lambda_2 \in \mathbb{C}. \tag{2.15}$$

Technical details on the proof of classical integrability are provided e.g. in [2, 11, 34].

2.1 Classical integrals of motion in ATFT

We shall exemplify our investigation using the first non-trivial model of the ATFT hierarchy that exhibits both types of boundary conditions, that is the $A_2^{(1)}$ case. Recall the Lax pair for a generic $A_n^{(1)}$ theory [8]:

$$\begin{aligned}
\mathbb{V}(x,t,u) &= \frac{\beta}{2}\,\partial_x\Phi\cdot H + \frac{m}{4}\,\left(u\ e^{\frac{\beta}{2}\Phi\cdot H}\ E_+\ e^{-\frac{\beta}{2}\Phi\cdot H} - \frac{1}{u}\ e^{-\frac{\beta}{2}\Phi\cdot H}\ E_-\ e^{\frac{\beta}{2}\Phi\cdot H}\right)\\
\mathbb{U}(x,t,u) &= \frac{\beta}{2}\,\Pi\cdot H + \frac{m}{4}\,\left(u\ e^{\frac{\beta}{2}\Phi\cdot H}\ E_+\ e^{-\frac{\beta}{2}\Phi\cdot H} + \frac{1}{u}\ e^{-\frac{\beta}{2}\Phi\cdot H}\ E_-\ e^{\frac{\beta}{2}\Phi\cdot H}\right)
\end{aligned} \tag{2.16}$$

$\Phi,\ \Pi$ are n-vector fields, with components $\phi_i,\ \pi_i,\ i\in\{1,\dots,n\},\ u = e^{\frac{2\lambda}{n+1}}$ is the multiplicative spectral parameter. To compare with the notation used in [11] we set $\frac{m^2}{16} = \frac{\tilde{m}^2}{8}$ ($\tilde{m}$ denotes the mass in [11]). Note that eventually in [11] both $\beta,\ \tilde{m}$ are set equal to unit.

Also define:

$$E_+ = \sum_{i=1}^{n+1} E_{\alpha_i}, \qquad E_- = \sum_{i=1}^{n+1} E_{-\alpha_i} \tag{2.17}$$

α_i are the simple roots, H (n-vector) and $E_{\pm\alpha_i}$ are the algebra generators in the Cartan-Weyl basis corresponding to simple roots, and they satisfy the Lie algebra relations:

$$\begin{aligned}
[H,\ E_{\pm\alpha_i}] &= \pm\alpha_i E_{\alpha_i},\\
[E_{\alpha_i},\ E_{-\alpha_i}] &= \frac{2}{\alpha_i^2}\ \alpha_i\cdot H
\end{aligned} \tag{2.18}$$

Explicit expressions on the simple roots and the Cartan generators are presented in appendix A. Notice that the Lax pair has the following behavior:

$$\mathbb{V}^t(x,t,-u^{-1}) = \mathbb{V}(x,t,u), \qquad \mathbb{U}^t(x,t,u^{-1}) = \mathbb{U}(x,t,u) \tag{2.19}$$

where ^{t} denotes usual transposition.

Our objective as mentioned is to examine the system with non-trivial boundaries, thus we consider representations of the associated underlying algebras expressed by $\mathcal{T}$. To recover the local integrals of motion of the considered system we shall follow the quite standard procedure and expand $\ln\ t(u)$ in powers of u^{-1}. An alternative strategy would be to derive the modified Lax pair, compatible with the boundary conditions chosen, and hence

the associated equations of motion (see e.g. [11]). A systematic derivation of boundary Lax pairs independently of the choice of model is discussed in [42]. To expand the open transfer matrix and derive the local integrals of motion we shall need the expansions of $T(x,y,u)$, $T(x,y,u^{-1})$ and $K^{\pm}(u)$. In what follows in the present section we basically introduce the necessary preliminaries for such a derivation, and we also reproduce the known integrals of motion for the ATFT on the full line.

Let $T'(x,y,u) = T(x,y,u^{-1})$ and $\mathbb{U}'(x,u) = \mathbb{U}(x,u^{-1})$. Following the logic described in [32] for the sine-Gordon model, we aim at expressing the part associated to E_+, E_- in $\mathbb{U}$, $\mathbb{U}'$ respectively independently of the fields, after applying a suitable gauge transformation. More precisely, consider the following gauge transformation:

$$\begin{aligned} &T(x,y,u) = \Omega(x)\ \tilde{T}(x,y,u)\ \Omega^{-1}(y), \\ &T'(x,y,u) = \Omega^{-1}(x)\ \tilde{T}'(x,y,u)\ \Omega(y) \qquad \Omega(x) = e^{\frac{\beta}{2}\Phi(x)\cdot H}. \end{aligned} \tag{2.20}$$

Then from equation (2.1) we obtain that the gauge transformed operators $\mathbb{U}$, $\mathbb{U}'$ can be expressed as:

$$\begin{aligned} \tilde{\mathbb{U}}(x,t,u) &= \Omega^{-1}(x)\ \mathbb{U}(x,t,u)\ \Omega(x) - \Omega^{-1}(x)\ \frac{d\Omega(x)}{dx} \\ \tilde{\mathbb{U}}'(x,t,u) &= \Omega(x)\ \mathbb{U}'(x,t,u)\ \Omega^{-1}(x) - \Omega(x)\ \frac{d\Omega^{-1}(x)}{dx}. \end{aligned} \tag{2.21}$$

After implementing the gauge transformations the operators $\tilde{\mathbb{U}}$, $\tilde{\mathbb{U}}'$ take the following simple form:

$$\begin{aligned} \tilde{\mathbb{U}}(x,t,u) &= \frac{\beta}{2}\Theta\cdot H + \frac{m}{4}\left(uE_+ + \frac{1}{u}X_-\right), \\ \tilde{\mathbb{U}}'(x,t,u) &= \frac{\beta}{2}\hat{\Theta}\cdot H + \frac{m}{4}\left(uE_- + \frac{1}{u}X_+\right) \end{aligned} \tag{2.22}$$

where we define:

$$\begin{aligned} \Theta &= \Pi - \partial_x\Phi, & \hat{\Theta} &= \Pi + \partial_x\Phi, \\ X_- &= e^{-\beta\Phi\cdot H}\ E_-\ e^{\beta\Phi\cdot H}, & X_+ &= e^{\beta\Phi\cdot H}\ E_+\ e^{-\beta\Phi\cdot H} \end{aligned} \tag{2.23}$$

$\tilde{T}$, $\tilde{\mathbb{U}}$ also satisfy (2.1), and Θ, $\hat{\Theta}$ are n vectors with components θ_i, $\hat{\theta}_i$ respectively.

Consider now the following ansatz for $\tilde{T}$, $\tilde{T}'$ as $|u| \to \infty$ [32]

$$\begin{aligned} \tilde{T}(x,y,u) &= (\mathbb{I} + W(x,u))\ \exp[Z(x,y,u)]\ (\mathbb{I} + W(y,u))^{-1}, \\ \tilde{T}'(x,y,u) &= (\mathbb{I} + \hat{W}(x,u))\ \exp[\hat{Z}(x,y,u)]\ (\mathbb{I} + \hat{W}(y,u))^{-1}, \end{aligned} \tag{2.24}$$

where W, $\hat{W}$ are off diagonal matrices i.e. $W = \sum_{i\neq j} W_{ij}E_{ij}$, and Z, $\hat{Z}$ are purely diagonal $Z = \sum_{i=1}^{n+1} Z_{ii}E_{ii}$. Also

$$Z(u) = \sum_{k=-1}^{\infty} \frac{Z^{(k)}}{u^k}, \qquad W_{ij} = \sum_{k=0}^{\infty} \frac{W^{(k)}}{u^k}. \tag{2.25}$$

Inserting the latter expressions (2.25) in (2.1) one may identify the coefficients $W_{ij}^{(k)}$ and $Z_{ii}^{(k)}$. Indeed from (2.1) we obtain the following fundamental relations:

$$\frac{dZ}{dx} = \tilde{\mathbb{U}}^{(D)} + (\tilde{\mathbb{U}}^{(O)}\ W)^{(O)}$$
$$\frac{dW}{dx} + W\tilde{\mathbb{U}}^{(D)} - \tilde{\mathbb{U}}^{(D)}W + W(\tilde{\mathbb{U}}^{(O)}W)^{(D)} - \tilde{\mathbb{U}}^{(O)} - (\tilde{\mathbb{U}}^{(O)}W)^{(O)} = 0 \tag{2.26}$$

where the superscripts $O,\ D$ denote off-diagonal and diagonal part respectively. Similar relations may be obtained for $\hat{Z},\ \hat{W}$, in this case $\tilde{\mathbb{U}} \to \tilde{\mathbb{U}}'$. We omit writing these equations here for brevity.

It will be useful in what follows to introduce some notation:

$$\frac{\beta}{2}\Theta \cdot H = \mathrm{diag}(a,\ b,\ c), \qquad \frac{\beta}{2}\hat{\Theta} \cdot H = \mathrm{diag}(\hat{a},\ \hat{b},\ \hat{c}), \qquad e^{\beta\alpha_i\cdot\Phi} = \gamma_i \tag{2.27}$$

explicit expression of $a,\ b,\ c$ and γ_i can be found in appendix B (B.4); notice that $a+b+c = 0$. From the first of equations (2.26) we may derive the matrices $Z,\ \hat{Z}$. Indeed one may easily show that:

$$\frac{dZ^{(0)}}{dx} = \frac{m}{4}\begin{pmatrix} W_{21}^{(1)} + \zeta a & & \\ & W_{32}^{(1)} + \zeta b & \\ & & -W_{13}^{(1)} + \zeta c \end{pmatrix} = 0$$
$$\frac{d\hat{Z}^{(0)}}{dx} = \frac{m}{4}\begin{pmatrix} -\hat{W}_{31}^{(1)} + \zeta\hat{a} & & \\ & \hat{W}_{12}^{(1)} + \zeta\hat{b} & \\ & & \hat{W}_{23}^{(1)} + \zeta\hat{c} \end{pmatrix} = 0 \tag{2.28}$$

it is clear that the latter quantities are zero because of the form of $W_{ij}^{(1)},\ \hat{W}_{ij}^{(1)}$ see appendix B. Also the higher order $Z^{(k)},\ \hat{Z}^{(k)}$ are given by:

$$\frac{dZ^{(k)}}{dx} = \frac{m}{4}\begin{pmatrix} W_{21}^{(k+1)} - \gamma_3 W_{31}^{(k-1)} & & \\ & W_{32}^{(k+1)} + \gamma_1 W_{12}^{(k-1)} & \\ & & -W_{13}^{(k+1)} + \gamma_2 W_{23}^{(k-1)} \end{pmatrix}$$
$$\frac{d\hat{Z}^{(k)}}{dx} = \frac{m}{4}\begin{pmatrix} -\hat{W}_{31}^{(k+1)} + \gamma_1 \hat{W}_{21}^{(k-1)} & & \\ & \hat{W}_{12}^{(k+1)} + \gamma_2 \hat{W}_{32}^{(k-1)} & \\ & & \hat{W}_{23}^{(k+1)} - \gamma_3 \hat{W}_{13}^{(k-1)} \end{pmatrix}$$
$$k > 0. \tag{2.29}$$

The computation of $W,\ \hat{W}$ is essential for defining the diagonal elements. First it is important to discuss the leading contribution of the above quantities as $|u| \to \infty$. To achieve this we shall need the explicit form of $Z^{(-1)},\ \hat{Z}^{(-1)}$:

$$Z^{(-1)}(x,y) = \frac{m(x-y)}{4}\begin{pmatrix} e^{\frac{i\pi}{3}} & & \\ & e^{-\frac{i\pi}{3}} & \\ & & -1 \end{pmatrix}, \quad \hat{Z}^{(-1)}(x,y) = \frac{m(x-y)}{4}\begin{pmatrix} e^{-\frac{i\pi}{3}} & & \\ & e^{\frac{i\pi}{3}} & \\ & & -1 \end{pmatrix}. \tag{2.30}$$

The information above will be extensively used in what follows.

Before we proceed with the analysis of integrable boundary conditions in ATFT let us first reproduce the known local integrals of motion in the periodic case, emerging from the expansion ($|u| \to \infty$)

$$\ln\,[trT(u)] = \ln\,[tr\{(1+W(L,u))\;e^{Z(L,-L,u)}\;(1+W(-L,u))^{-1}\}]. \tag{2.31}$$

Notice that in the case of periodic boundary conditions we put our system in the 'whole' line ($x=L,\ y=-L$), and consider Schwartz boundary conditions, i.e. the fields and their derivatives vanish at the end points $\pm L$. Bearing in mind that as $u \to -\infty$ the leading contribution of $e^{Z},\ (e^{\hat{Z}})$ (see (2.30)) comes from the $e^{Z_{33}},\ (e^{\hat{Z}_{33}})$ term, the expression above becomes

$$\ln\,[trT(u\to-\infty)] = \sum_{k=-1}\frac{Z_{33}^{(k)}}{u^k}. \tag{2.32}$$

To reproduce the familiar local integrals of motion we shall need both $Z(L,-L,u),\ \hat{Z}(L,-L,u)$. Let

$$\begin{aligned}
\mathcal{I}_1 &= -\frac{12m}{\beta^2}Z_{33}^{(1)}(L,-L,u) = \int_{-L}^{L}dx\left(\sum_{i=1}^{2}\theta_i^2+\frac{m^2}{\beta^2}\sum_{i=1}^{3}e^{\beta\alpha_i\cdot\Phi}\right),\\
\mathcal{I}_{-1} &= -\frac{12m}{\beta^2}\hat{Z}_{33}^{(1)}(L,-L,u) = \int_{-L}^{L}dx\left(\sum_{i=1}^{2}\hat{\theta}_i^2+\frac{m^2}{\beta^2}\sum_{i=1}^{3}e^{\beta\alpha_i\cdot\Phi}\right)\\
\mathcal{I}_2 &= \frac{3m^2}{2\beta^3}Z_{33}^{(2)}(L,-L,u) = \int_{-L}^{L}dx\left(\frac{8}{\beta^3}\left(abc-bc'\right)-\frac{m^2}{2\beta^3}\left(\gamma_1 c+\gamma_2 a+\gamma_3 b\right)\right)\\
\mathcal{I}_{-2} &= \frac{3m^2}{2\beta^3}\hat{Z}_{33}^{(2)}(L,-L,u) = \int_{-L}^{L}dx\left(\frac{8}{\beta^3}\left(\hat{a}\hat{b}\hat{c}+\hat{b}\hat{c}'\right)-\frac{m^2}{2\beta^3}\left(\gamma_1\hat{c}+\gamma_2\hat{a}+\gamma_3\hat{b}\right)\right)\\
&\ldots\quad\text{(higher local integrals of motion)}
\end{aligned} \tag{2.33}$$

the momentum and Hamiltonian (and the higher conserved quantities) of the ATFT are given by:

$$\begin{aligned}
\mathcal{P}_1 &= \frac{1}{2}(\mathcal{I}_{-1}-\mathcal{I}_1) = \int_{-L}^{L}dx\sum_{i=1}^{2}\left(\pi_i\;\phi_i'-\pi_i'\;\phi_i\right)\\
\mathcal{H}_1 &= \frac{1}{2}(\mathcal{I}_1+\mathcal{I}_{-1}) = \int_{-L}^{L}dx\left(\sum_{i=1}^{2}(\pi_i^2+\phi_i'^2)+\frac{m^2}{\beta^2}\sum_{i=1}^{3}e^{\beta\alpha_i\cdot\Phi}\right)\\
\mathcal{P}_2 &= \frac{1}{2}(\mathcal{I}_{-2}-\mathcal{I}_2)\\
&= \frac{1}{2}\int_{-L}^{L}dx\left(\frac{8}{\beta^3}\left(\hat{a}\hat{b}\hat{c}-abc\right)+\frac{8}{\beta^3}\left(bc'+\hat{b}\hat{c}'\right)+\frac{m^2}{2\beta^3}\left(\gamma_1\left(c-\hat{c}\right)+\gamma_2\left(a-\hat{a}\right)+\gamma_3\left(b-\hat{b}\right)\right)\right)\\
\mathcal{H}_2 &= \frac{1}{2}(\mathcal{I}_2+\mathcal{I}_{-2})\\
&= \frac{1}{2}\int_{-L}^{L}dx\left(\frac{8}{\beta^3}\left(abc+\hat{a}\hat{b}\hat{c}\right)-\frac{8}{\beta^3}\left(bc'-\hat{b}\hat{c}'\right)-\frac{m^2}{2\beta^3}\left(\gamma_1\left(c+\hat{c}\right)+\gamma_2\left(a+\hat{a}\right)+\gamma_3\left(b+\hat{b}\right)\right)\right)\\
&\ldots
\end{aligned} \tag{2.34}$$

Note that the boundary terms are absent in the expressions above, since we considered Schwartz type boundary conditions. Also, in the generic situation, for any $A_n^{(1)}$, the sum in the momentum $\mathcal{P}_1$ and the kinetic term of the Hamiltonian $\mathcal{H}_1$ runs from 1 to n, whereas the sum in the potential term of the Hamiltonian runs from 1 to $n+1$.

3. SNP boundary conditions

We turn now to our main concern, which is the study of integrable boundary conditions in ATFT. We shall first discuss the boundary conditions that already have been analyzed in [11]. Based on the underlying algebra, that is the classical analogue of the q-twisted Yangian we shall reproduce the previously known results [11], so this section serves basically as a warm up exercise. In the subsequent section we shall analyze in detail the novel boundary conditions (SP) associated to the classical version of the reflection algebra.

To obtain the relevant local integrals of motion we shall expand the following object (consider now $x=0,\ y=-L$):

$$\begin{aligned}\ln\ t(u) &= \ln\ tr\Big\{K^+(u)\ T(u)\ K^-(u)\ T^t(u^{-1})\Big\}\\ &= \ln\ tr\Big\{K^+(u)\ \Omega(0)\ \tilde{T}(u)\ \Omega^{-1}(-L)\ K^-(u)\ \Omega(-L)\ \tilde{T}^t(u^{-1})\ \Omega^{-1}(0)\Big\}\end{aligned} \quad (3.1)$$

For simplicity here, but without really losing generality we consider Schwartz boundary conditions at the boundary point $-L$ and $K^-(u)\propto\mathbb{I}$. Also $K^+(u)=K^t(u^{-1})$ where K is any c-number solution of the twisted Yangian. Taking also into account the ansatz (2.24) we conclude

$$\ln t(u)=\ln tr\left\{(1+\hat{W}^t(0,u))\Omega^{-1}(0)K^+(u)\Omega(0)(1+W(0,u))e^{Z(0,-L,u)+\hat{Z}(0,-L,u)}\right\}. \quad (3.2)$$

Recall from the previous section that as $u\to-\infty$ the leading contribution of $e^Z,\ e^{\hat{Z}}$ comes from the $e^{Z_{33}},\ e^{\hat{Z}_{33}}$ terms (see (2.30)), hence

$$\begin{aligned}\ln t(u)&=Z_{33}(0,-L,u)+\hat{Z}_{33}(0,-L,u)+\ln[(1+\hat{W}^t(0,u))\Omega^{-1}(0)K^+(u)\Omega(0)(1+W(0,u))]_{33}\\ &=\sum_{k=-1}^{\infty}\frac{Z_{33}^{(k)}+\hat{Z}_{33}^{(k)}}{u^k}+\sum_{k=0}^{\infty}\frac{\mathrm{f}_k}{u^k}.\end{aligned} \quad (3.3)$$

To obtain the explicit form of the boundary contributions to the integrals of motion we should first review known results on the solution of the reflection equation for SNP boundary conditions. The generic solution for the $A_n^{(1)}$ case in the principal gradation are given by [17, 22]:

$$\begin{aligned}K(\lambda)\ &=\ \Big(ge^{\lambda}+\bar{g}e^{-\lambda}\Big)\sum_{i=1}^{n+1}e_{ii}+\sum_{i>j}f_{ij}e^{\lambda-\frac{2\lambda}{n+1}(i-j)}e_{ij}+\sum_{i<j}f_{ij}e^{-\lambda-\frac{2\lambda}{n+1}(i-j)}e_{ij}\\ g\ &=\ q^{-\frac{1}{2}+\frac{n+1}{4}}\qquad \bar{g}=\pm q^{\frac{1}{2}-\frac{n+1}{4}},\quad f_{ij}=\pm q^{-\frac{n+1}{4}},\qquad f_{ji}=q^{\frac{n+1}{4}},\quad i<j.\end{aligned} \quad (3.4)$$

In order to effectively compare with the results of [17] as well as being compatible with [11] we always express in the text both r and K matrices in the principal gradation. Nevertheless, to obtain the matrix in the homogenous gradation as given in [22] we implement the

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following gauge transformation

$$K^{(h)} = \mathcal{V}(\lambda)\ K^{(p)}(\lambda)\ \mathcal{V}(-\lambda). \tag{3.5}$$

We shall now focus on the $A_2^{(1)}$ case, which is our main example here. Recall that $K^+(u) = K^t(u^{-1})$ then the K^+-matrix is 3×3 matrix written explicitly as:

$$\begin{aligned} K^+(u) &= u^{\frac{3}{2}}\bar{G} + u^{\frac{1}{2}}\bar{F} + u^{-\frac{1}{2}}F + u^{-\frac{3}{2}}G \qquad\qquad \text{where} \\ G &= g\ \mathbb{I}, \qquad\qquad \bar{G} = \bar{g}\ \mathbb{I}, \\ \bar{F} &= f_{12}\ e_{21} + f_{23}\ e_{32} + f_{31}\ e_{13}, \\ F &= f_{21}\ e_{12} + f_{32}\ e_{23} + f_{13}\ e_{31} \end{aligned} \tag{3.6}$$

and the coefficients $g,\ \bar{g},\ f_{ij}$ are given in (3.4) with $n=2$. Bearing in mind the explicit form of the boundary matrix we may identify the factors f_i in the expansion (3.3) which are reported in appendix C. Taken into account expressions (3.3), (C.1) and $Z_{33}^{(1)},\ \hat{Z}_{33}^{(1)}$ given in appendix B we conclude for the first non-trivial boundary integral of motion:

$$\begin{aligned} \mathcal{H}_1^{(b)} &= -\frac{6m}{\beta^2}\left(Z_{33}^{(1)} + \hat{Z}_{33}^{(1)} + \mathrm{f}_1\right) \\ &= \int_{-L}^{0} dx \left(\sum_{i=1}^{2}\left(\pi_i^2 + \phi_i'^2\right) + \frac{m^2}{\beta^2}\sum_{i=1}^{3} e^{\beta\alpha_i\cdot\Phi}\right) \\ &\qquad + \frac{2m}{\bar{g}\beta^2}\left(f_{12}e^{\frac{\beta}{2}\alpha_1\cdot\Phi(0)} + f_{23}e^{\frac{\beta}{2}\alpha_2\cdot\Phi(0)} - f_{31}e^{\frac{\beta}{2}\alpha_3\cdot\Phi(0)}\right). \end{aligned} \tag{3.7}$$

In general for the $A_n^{(1)}$ ATFT the boundary Hamiltonian with SNP boundary conditions will have the following from

$$\mathcal{H}_1^{(b)} = \int_{-L}^{0} dx \left(\sum_{i=1}^{n}(\pi_i^2 + \phi_i'^2) + \frac{m^2}{\beta^2}\sum_{i=1}^{n+1} e^{\beta\alpha_i\cdot\Phi}\right) + \sum_{i=1}^{n+1} c_i\ e^{\frac{\beta}{2}\alpha_i\cdot\Phi(0)}, \tag{3.8}$$

which as expected coincides with the boundary Hamiltonian deduced in [11]. It is quite easy to check that in the case of a trivial boundary conditions, i.e. $K^+ \propto \mathbb{I}$ the boundary terms containing c_i disappear and the entailed Hamiltonian has exactly the same structure as in the bulk case.

The second conserved charge of the hierarchy is given by

$$\begin{aligned} \mathcal{H}_2^{(b)} &= \frac{3m^2}{4\beta^3}\left(Z_{33}^{(2)} + \hat{Z}_{33}^{(2)} + \mathrm{f}_2\right) \\ &= \frac{1}{2}\int_{-L}^{0} dx \left(\frac{8}{\beta^3}\left(abc + \hat{a}\hat{b}\hat{c}\right) - \frac{8}{\beta^3}\left(bc' - \hat{b}\hat{c}'\right) - \frac{m^2}{2\beta^3}\left(\gamma_1(c+\hat{c}) + \gamma_2(a+\hat{a}) + \gamma_3\left(b+\hat{b}\right)\right)\right) \\ &\quad - \frac{m^2}{4\bar{g}\beta^2}\left(f_{21}e^{-\frac{\beta}{2}\alpha_1\cdot\Phi(0)} + f_{32}e^{-\frac{\beta}{2}\alpha_2\cdot\Phi(0)} - f_{13}e^{-\frac{\beta}{2}\alpha_3\cdot\Phi(0)}\right) + \frac{4}{\beta^2}\left(\hat{c}^2(0) - \hat{a}(0)c(0) - b(0)\hat{c}(0)\right) \\ &\quad - \frac{m}{\bar{g}\beta^2}\left(f_{12}(c(0)+\hat{c}(0))e^{\frac{\beta}{2}\alpha_1\cdot\Phi(0)} - f_{23}b(0)e^{\frac{\beta}{2}\alpha_2\cdot\Phi(0)} + f_{31}\hat{a}(0)e^{\frac{\beta}{2}\alpha_3\cdot\Phi(0)}\right) \\ &\quad - \frac{3m^2}{8\bar{g}\beta^2}\left(-\frac{1}{3\bar{g}}\left(f_{12}e^{\frac{\beta}{2}\alpha_1\cdot\Phi(0)} + f_{23}e^{\frac{\beta}{2}\alpha_2\cdot\Phi(0)} - f_{31}e^{\frac{\beta}{2}\alpha_3\cdot\Phi(0)}\right) + \frac{\zeta}{3}\left(c(0) - b(0) + \hat{c}(0) - \hat{a}(0)\right)\right)^2. \end{aligned} \tag{3.9}$$

Again when assuming the simplest boundary conditions $K^+ \propto \mathbb{I}$ we conclude that all the boundary terms containing the factors f_{ij} disappear, exactly as it happens in the first Hamiltonian. The bulk parts of the boundary Hamiltonians above coincide with the ones found in the previous section –for Schwartz type boundary conditions. Extra boundary terms are added due to the presence of the non-trivial K-matrix.

Notice that the boundary analogues of $\mathcal{P}_k$ are not conserved quantities anymore similarly to the sine-Gordon model on the half line, where only the 'half' of the bulk charges are conserved after the implementation of consistent integrable boundary conditions. We should stress that this is a consequence of the particular choice of boundary conditions, and this will become apparent in the next section while analyzing the novel boundary conditions. Note also that in the expressions for the boundary Hamiltonian written above there exist no free boundary parameter, contrary to the SP case as will see subsequently. Analogous results may be seen in the context of quantum integrable spin chains regarding the explicit expression of the Hamiltonian as well as the symmetry of the relevant models [31, 21, 22].

Let us finally mention that one can in general consider 'dynamical' boundary conditions (see e.g. [12, 43, 34]). In this case instead of assuming a c-number solution of the classical version of the q-twisted Yangian (2.11) we consider a generic –dynamical– representation of the algebra defined as [2]:

$$\mathbb{K}(\lambda) = L(\lambda - \Theta)\ K(\lambda) \otimes \mathbb{I}\ L^t(-\lambda - \Theta) \tag{3.10}$$

where K is a c-number solution of the classical twisted Yangian [17, 22], and L is any solution of the fundamental relation (2.6) e.g. a q-oscillator. Such boundary conditions for the $A_2^{(1)}$ ATFT have been analyzed in [12]. More precisely, in this case the entries of $\mathbb{K}$ are not c-number anymore, but algebraic objects satisfying Poisson commutation relations dictated by the underlying classical algebra. At the quantum level these objects, and consequently the quantities f_{ij} appearing in the local integrals of motion (3.7), (3.9), become operators obeying commutation relation defined by the q-twisted Yangian. In fact, due to the 'dynamical nature' of the boundary conditions extra degrees of freedom, incorporated in L, are attached to the boundary.

4. SP boundary conditions

We come now to the study of the more intriguing, at least in the present context, boundary conditions. Here for the first time we systematically analyze the new boundary conditions (SP) starting from the underlying algebra i.e. the reflection algebra. In this case the generating function of the local integrals of motion is given by the following expression:

$$\begin{aligned} \ln\ t(u) &= \ln\ tr\left\{K^+(u)\ T(u)\ K^-(u)\ T^{-1}(u^{-1})\right\} \\ &= \ln\ tr\left\{K^+(u)\ \Omega(0)\ \tilde{T}(u)\ \Omega^{-1}(-L)\ K^-(u)\ \Omega^{-1}(-L)\ \tilde{T}^{-1}(u^{-1})\ \Omega(0)\right\} \end{aligned} \tag{4.1}$$

taking into account the ansatz (2.24) we conclude

$$\ln t(u) = \ln tr\Big\{(1+\hat{W}(0,u))^{-1}\Omega(0)K^{+}(u)\Omega(0)(1+W(0,u))e^{Z(0,-L,u)} \\ (1+W(-L,u))^{-1}\Omega^{-1}(-L)K^{-}(u)\Omega^{-1}(-L)e^{-\hat{Z}(0,-L,u)}(1+\hat{W}(-L,u))\Big\}. \tag{4.2}$$

The leading contribution of $e^{Z},\ e^{-\hat{Z}}$ comes from the $e^{Z_{11}},\ e^{-\hat{Z}_{11}}$ terms as $iu \to \infty$, whereas as $iu \to -\infty$ it comes from the $e^{Z_{22}},\ e^{-\hat{Z}_{22}}$ terms. Depending on the limit we assume we obtain two distinct expressions for $iu \to \infty$ and $iu \to -\infty$ respectively:

$$\begin{aligned}
\ln\ t(iu \to \infty) &= Z_{11}(0,-L,u) - \hat{Z}_{11}(0,-L,u) \\
&+ \ln[(1+\hat{W}(0,u))^{-1}\Omega(0)\ K^{+}(u)\ \Omega(0)\ (1+W(0,u))]_{11} \\
&+ \ln[(1+W(-L,u))^{-1}\Omega^{-1}(-L)\ K^{-}(u)\ \Omega^{-1}(-L)\ (1+\hat{W}(-L,u))]_{11} \\
\ln\ t(iu \to -\infty) &= Z_{22}(0,-L,u) - \hat{Z}_{22}(0,-L,u) \\
&+ \ln[(1+\hat{W}(0,u))^{-1}\Omega(0)\ K^{+}(u)\ \Omega(0)\ (1+W(0,u))]_{22} \\
&+ \ln[(1+W(-L,u))^{-1}\Omega^{-1}(-L)\ K^{-}(u)\ \Omega^{-1}(-L)\ (1+\hat{W}(-L,u))]_{22}.
\end{aligned} \tag{4.3}$$

Expanding all the terms above we get

$$\begin{aligned}
\ln\ t(iu \to \infty) &= \sum_{k=-1}^{\infty} \frac{Z_{11}^{(k)} - \hat{Z}_{11}^{(k)}}{u^n} + \sum_{k=0}^{\infty} \frac{\mathrm{f}_k^{+} + \mathrm{f}_k^{-}}{u^k} \\
\ln\ t(iu \to -\infty) &= \sum_{k=-1}^{\infty} \frac{Z_{22}^{(k)} - \hat{Z}_{22}^{(k)}}{u^k} + \sum_{k=0}^{\infty} \frac{\mathrm{h}_k^{+} + \mathrm{h}_k^{-}}{u^k}.
\end{aligned} \tag{4.4}$$

Although we follow exactly the same analysis as in the SNP case, we see that the investigation of the SP boundary conditions is technically more involved mainly due to the fact that one has to consider the behavior of the transfer matrix for both $iu \to \infty$ and $-iu \to \infty$. Another technically intriguing point is that the behavior of $(1+W)^{-1}$, which is quite intricate, must be studied even if the system is considered on the half line i.e. Schwartz type boundary conditions are set at the boundary point $-L$ (see for instance the previous section).

We shall focus here for simplicity only on diagonal solutions of the reflection equation [26] given by the following expressions (in the principal gradation):

$$K_{(l)}(\lambda,\xi) = \sinh(\lambda+i\xi)e^{-\lambda}\sum_{j=1}^{l} e^{-\frac{4\lambda}{n+1}(j-1)}e_{jj} + \sinh(-\lambda+i\xi)e^{\lambda}\sum_{j=l+1}^{n} e^{-\frac{4\lambda}{n+1}(j-1)}e_{jj} \tag{4.5}$$

(recall $u = e^{\frac{2\lambda}{n+1}}$). To obtain the K-matrix in the homogeneous gradation we implement a gauge transformation:

$$K_{(l)}^{(h)}(\lambda,\ \xi) = \mathcal{V}(\lambda)\ K_{(l)}^{(p)}(\lambda,\ \xi)\ \mathcal{V}(\lambda). \tag{4.6}$$

In fact, the presence of non-diagonal boundary conditions does not modify the structure of the local integrals of motion, but simply gives rise to more complicated boundary terms.

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Note that in the $A_2^{(1)}$ case we end up with two types of diagonal boundary matrices corresponding to the two possible values $l = 1,\ 2$. We shall consider an example here to demonstrate how the particular choice of boundary K-matrix contributes to the integrals of motion. Specifically, to obtain the most general results with the least effort it is practical to consider a non-trivial left boundary described by $K_{(1)}$, and a right boundary described by the $K_{(2)}$-matrix i.e.

$$K^+(u,\ \xi^+) = K_{(1)}(u^{-1},\ \xi^+), \qquad K^-(u,\ \xi^-) = K_{(2)}(u,\ \xi^-) \tag{4.7}$$

The integrals of motion emerging from the first order of the asymptotics of the transfer matrix as $iu \to \pm\infty$ are given by:

$$\mathbb{I}_1 = Z_{11}^{(1)} - \hat{Z}_{11}^{(1)} + \mathrm{f}_1^+ + \mathrm{f}_1^-, \quad \tilde{\mathbb{I}}_1 = Z_{22}^{(1)} - \hat{Z}_{22}^{(1)} + \mathrm{h}_1^+ + \mathrm{h}_1^- \tag{4.8}$$

(expressions on $Z,\ \hat{Z}$ are provided in appendix B). From the latter expressions (4.8) and gathering information from (C.3) and appendix B we obtain

$$\begin{aligned}
\mathbb{I}_1 =& \frac{m}{24}\Bigg(-\beta\alpha_2\cdot\Pi(0) - \frac{24}{m}e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)} + \beta\alpha_1\cdot\Pi(-L) - \frac{24}{m}e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)} \\
&\quad -\frac{\beta^2}{4}\int_{-L}^{0}dx\sum_{i=1}^{2}\pi_i\ \phi_i'\Bigg) \\
&-\frac{i\sqrt{3}m}{24}\Bigg(\beta\alpha_2\cdot\Phi'(0) - \beta\alpha_1\cdot\Phi'(-L) + \frac{m^2}{8}\int_{-L}^{0}dx\sum_{i=1}^{3}e^{\frac{\beta}{2}\alpha_i\cdot\Phi} + \frac{\beta^2}{8}\int_{-L}^{0}dx\sum_{i=1}^{2}\left(\pi_i^2+\phi_i'^2\right)\Bigg) \\
\tilde{\mathbb{I}}_1 =& \frac{m}{24}\Bigg(-\beta\alpha_2\cdot\Pi(0) - \frac{24}{m}e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)} + \beta\alpha_1\cdot\Pi(-L) - \frac{24}{m}e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)} \\
&\quad -\frac{\beta^2}{4}\int_{-L}^{0}dx\sum_{i=1}^{2}\pi_i\ \phi_i'\Bigg) \\
&+\frac{i\sqrt{3}m}{24}\Bigg(\beta\alpha_2\cdot\Phi'(0) - \beta\alpha_1\cdot\Phi'(-L) + \frac{m^2}{8}\int_{-L}^{0}dx\sum_{i=1}^{3}e^{\frac{\beta}{2}\alpha_i\cdot\Phi} + \frac{\beta^2}{8}\int_{-L}^{0}dx\sum_{i=1}^{2}\left(\pi_i^2+\phi_i'^2\right)\Bigg).
\end{aligned} \tag{4.9}$$

The momentum and energy are directly obtained from the above conserved quantities as:

$$\begin{aligned}
\mathcal{P}_1^{(b)} &= -\frac{6m}{\beta^2}\left(\mathbb{I}_1 + \tilde{\mathbb{I}}_1\right) \\
&= \int_{-L}^{0}dx\sum_{i=1}^{2}\left(\pi_i\ \phi_i' - \pi_i'\ \phi_i\right) + \sum_{i=1}^{2}\pi_i(0)\ \phi_i(0) + \frac{8}{\beta}\alpha_2\cdot\Pi(0) + \frac{12m}{\beta^2}e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)} \\
&\quad -\sum_{i=1}^{2}\pi_i(-L)\ \phi_i(-L) - \frac{8}{\beta}\alpha_1\cdot\Pi(-L) + \frac{12m}{\beta^2}e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)} \\
\mathcal{H}_1^{(b)} &= \frac{6im}{\sqrt{3}\beta^2}\left(\mathbb{I}_1 - \tilde{\mathbb{I}}_1\right) \\
&= \int_{-L}^{0}dx\left(\sum_{i=1}^{2}\left(\pi_i^2+\phi_i'^2\right) + \frac{m^2}{\beta^2}\sum_{i=1}^{3}e^{\beta\alpha_i\cdot\Phi}\right) + \frac{8}{\beta}\alpha_2\cdot\Phi'(0) - \frac{8}{\beta}\alpha_1\cdot\Phi'(-L).
\end{aligned} \tag{4.10}$$

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Notice the presence of the free boundary parameters $\xi^{\pm 3}$ in the local integrals of motion above, as opposed to the SNP case where no free boundary parameters appear in the corresponding integrals of motion. Naturally the two boundary cases are qualitatively distinguished; in SNP the c-number K-matrix contains no free parameters, and consequently no free parameters occur in the entailed integrals of motion. In the SP case however the K-matrix contains free parameters, which explicitly appear in the boundary integrals of motion. The implementation of non-diagonal K-matrices would lead to the appearance of extra boundary terms and parameters in the induced local integrals of motion.

The integrals of motion emerging from the second order of the expansion are derived as:

$$\begin{aligned}\mathbb{I}_2 &= Z_{11}^{(2)} - \hat{Z}_{11}^{(2)} + \mathrm{f}_2^+ + \mathrm{f}_2^- = \frac{4\beta^3}{3m^2}\left(\mathcal{P}_2^{(b)} + i\sqrt{3}\mathcal{H}_2^{(b)}\right), \\ \tilde{\mathbb{I}}_2 &= Z_{22}^{(2)} - \hat{Z}_{22}^{(2)} + \mathrm{h}_2^+ + \mathrm{h}_2^- = \frac{4\beta^3}{3m^2}\left(\mathcal{P}_2^{(b)} - i\sqrt{3}\mathcal{H}_2^{(b)}\right)\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}\mathcal{P}_2^{(b)} = \frac{1}{2}\int_{-L}^{0} dx \Big(&\frac{8}{\beta^3}\left(\hat{a}\hat{b}\hat{c} - abc\right) + \frac{8}{\beta^3}\left(bc' + \hat{b}\hat{c}'\right) + \frac{m^2}{2\beta^3}\left(\gamma_1\left(c - \hat{c}\right) + \gamma_2\left(a - \hat{a}\right) + \gamma_3\left(b - \hat{b}\right)\right)\Big) \\ &+ \frac{m^2}{4\beta^3}\left(\gamma_1(0) - \gamma_2(0)\right) + \frac{3m^2}{4\beta^3}e^{\beta\alpha_2\cdot\Phi(0)} - \frac{3m^2}{8\beta^3}e^{-4i\xi^+}e^{-2\beta\alpha_3\cdot\Phi(0)} \\ &+ \frac{3m}{2\beta^3}e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)}\left(c(0) + \hat{c}(0)\right) + \frac{2}{\beta^3}\left(\hat{b}'(0) - b'(0)\right) + \frac{2}{\beta^3}\left(b^2(0) + \hat{b}^2(0)\right) \\ &+ \frac{m^2}{4\beta^3}\left(\gamma_2(-L) - \gamma_1(-L)\right) + \frac{3m^2}{4\beta^3}e^{\beta\alpha_1\cdot\Phi(-L)} - \frac{3m^2}{8\beta^3}e^{-4i\xi^-}e^{-2\beta\alpha_3\cdot\Phi(-L)} \\ &+ \frac{3m}{2\beta^3}e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)}\left(a(-L) + \hat{a}(-L)\right) \\ &+ \frac{2}{\beta^3}\left(b'(-L) - \hat{b}'(-L)\right) + \frac{1}{\beta^3}\left(\hat{b}^2(-L) + b^2(-L)\right)\end{aligned} \tag{4.12}$$

$$\begin{aligned}\mathcal{H}_2^{(b)} = \frac{1}{2}\int_{-L}^{L} dx \Big(&\frac{8}{\beta^3}\left(abc + \hat{a}\hat{b}\hat{c}\right) - \frac{8}{\beta^3}\left(bc' - \hat{b}\hat{c}'\right) - \frac{m^2}{2\beta^3}\left(\gamma_1\left(c + \hat{c}\right) + \gamma_2\left(a + \hat{a}\right) + \gamma_3\left(b + \hat{b}\right)\right)\Big) \\ &+ \frac{3m}{2\beta^3}e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)}\left(\hat{c}(0) - c(0)\right) - \frac{2}{\beta^3}\left(b'(0) + \hat{b}'(0)\right) + \frac{2}{\beta^3}\left(b^2(0) - \hat{b}^2(0)\right) \\ &+ \frac{3m}{2\beta^3}e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)}\left(\hat{a}(-L) - a(-L)\right) \\ &+ \frac{2}{\beta^3}\left(b'(-L) + \hat{b}'(-L)\right) + \frac{1}{\beta^3}\left(b^2(-L) - \hat{b}^2(-L)\right).\end{aligned} \tag{4.13}$$

Higher integrals of motion are naturally obtained from the higher order expansion of the open transfer matrix but we shall not further pursue this point here. Notice that both $\mathcal{H}_k^{(b)}$ and $\mathcal{P}_k^{(b)}$ are conserved quantities contrary to what happens in the SNP case analyzed in the previous section, where only $\mathcal{H}_k^{(b)}$ are conserved. This is another basic qualitative difference between the two types of boundary conditions. Note that from the deduced

[3]The parameters ξ^+, ξ^- are associated to the right left boundary respectively. Note also that there is an implicit dependence on the integers $l^\pm$.

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integrals of motion certain sets of equations of motion are entailed. In particular the equations of motion arise from the following equations:

$$\frac{\partial\phi_i(x,t)}{\partial t}=\Big\{\mathcal{H}_1^{(b)}(0,-L),\ \phi_i(x,t)\Big\}\,,\qquad \frac{\partial\pi_i(x,t)}{\partial t}=\Big\{\mathcal{H}_1^{(b)}(0,-L),\ \pi_i(x,t)\Big\}\,,$$
$$-L\leq x\leq 0,\qquad\qquad i\in\{1,\ldots,n\}. \tag{4.14}$$

A detailed discussion on the associated equations of motion and the relevant boundary Lax pairs systematically constructed along the lines described in [42] will be presented in a forthcoming publication.

As in the analysis of the preceding section for the classical twisted Yangian (SNP) we may as well consider dynamical boundary conditions in the SP case. Specifically, one can assume a generic –dynamical– representation of the underlying classical reflection algebra (2.11) defined as [2]:

$$\mathbb{K}(\lambda)=L(\lambda-\Theta)\ K(\lambda)\otimes\mathbb{I}\ L^{-1}(-\lambda-\Theta) \tag{4.15}$$

where K is a c-number solution of the classical reflection algebra [26], and L is any solution of (2.6). Again the extra boundary degrees of freedom are incorporated in L. A more detailed analysis of such boundary conditions in the ATFT frame will be presented elsewhere (see similar analysis for the sine-Gordon and the vector NLS models in [43] and [34] respectively).

5. Discussion

An exhaustive study of the integrable boundary conditions in $A_n^{(1)}$ ATFT was presented by systematically deriving the associated local integrals of motion. The key point in our analysis is the extraction of the local integrals of motion directly from the transfer matrix asymptotic expansion, and there is no conjecture involved as far as their structure is concerned. The systematic derivation of the boundary integrals of motion starting from the underlying algebra gives rise to two distinct types of boundary conditions associated to the reflection algebra and q-twisted Yangian.

Noticeably the SP boundary conditions are absent in the analysis presented in [11] mainly because of the a priori strong constraints imposed upon the structure of the boundary conserved local quantities. In [11] quantities of the type $\mathcal{P}_k$ were a priori disregarded as non conserved –this is true however only for the sine(sinh)-Gordon model ($A_1^{(1)}$)– whereas as we see in the present investigation these objects play a key role in distinguishing the two types of boundary conditions! Although sine-Gordon is the prototype model of the class under consideration an 'imitation' of its boundary behavior by the higher members of the hierarchy could be quite misleading. This is primarily due to the fact that the sine-Gordon is a self-conjugate model –soliton and anti-soliton are equivalent entities– and as such it has a very peculiar boundary behavior that cannot be naively generalized to higher $A_n^{(1)}$ ATFT.

One of the basic differences between the two types of boundary conditions is that in the SP case the number of integrals of motion is 'double' compared to the SNP ones. This phenomenon not only indicates a qualitatively different behavior of the model as far as the

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boundaries are concerned, but also leads to a modification of the bulk behavior altogether (see also e.g. [34]). The 'duplication' of the local integrals of motion in the SP case seems to persist to higher orders –we checked explicitly up to third order. More precisely, let $\mathcal{Q}_k,\ \mathcal{Q}_{-k}$ be the local integrals of motion of the $A_n^{(1)}$ ATFT on the full line, then the boundary conserved quantities for each type of boundary conditions are provided by:

$$\begin{aligned} \mathcal{Q}_k^{(b)} &= \mathcal{Q}_k + \mathcal{Q}_{-k} + \mathcal{B}_k && \text{for SNP} \\ \mathcal{Q}_k^{\pm(b)} &= \mathcal{Q}_k \pm \mathcal{Q}_{-k} + \mathcal{B}_k^{\pm} && \text{for SP} \end{aligned} \tag{5.1}$$

$\mathcal{B}_k,\ \mathcal{B}_k^{\pm}$ are the relevant boundary terms. In the SNP case only the integrals of motion where the bulk part is provided by the sum of $\mathcal{Q}_k,\ \mathcal{Q}_{-k}$ survive, while in SNP both sums and differences provide local conserved quantities, i.e. each one of $\mathcal{Q}_{\pm k}$ (with appropriate boundary terms) is a conserved quantity. Moreover in the SNP case no free parameters appear in the integrals of motion due to fact that the corresponding c-number K-matrices contain no free parameters. However in the SP case, as anticipated, the relevant integrals of motion depend on the parameters $\xi^{\pm},\ l^{\pm}$. It is worth stressing that in the context of integrable spin chains the parameters $\xi^{\pm},\ l^{\pm}$ explicitly appear in the corresponding Hamiltonian as well as in the associated symmetry of the model. More precisely, it was shown in [28] that the rational open spin chain with diagonal boundary conditions associated to integers $l^{\pm} = l$ is $gl_l \otimes gl_{n+1-l}$ invariant and $\mathcal{U}_q(gl_l) \otimes \mathcal{U}_q(gl_{n+1-l})$ invariant in the trigonometric case, relevant to the ATFT theories. Recall that the $\mathcal{U}_q(gl_{n+1})$ spin chain maybe thought of as an integrable lattice version of the $A_n^{(1)}$ ATFT in the same logic that the critical XXZ spin chain may be seen as the lattice version of the sine-Gordon model.

There exist various studies concerning the underlying symmetry algebras when non-trivial integrable boundary conditions are present. Specifically, the symmetry algebra in the context of ATFT with SNP boundary conditions –being a twisted algebra– was investigated in [18], while extensive studies on the underlying algebras in integrable spin chains with both types of boundary conditions are presented in [23, 24]. An analysis in the spirit of [18, 44] would provide the non-local integrals of motion forming the exact symmetry algebra in the SP case, however this will be presented in a separate publication (see a relevant analysis in the quantum case in [23]).

Another intriguing point associated to the 'folding' of integrals of motion is the possible folding of the classical counterparts of Bethe ansatz equations in the SNP case emerging from the solution of the spectral problem [32, 45]. Although folding of Bethe anastz equations has been reported so far only in isotropic examples we conjecture that it should also occur in models associated to trigonometric R-matrices. In general the structure of Bethe ansatz is immediately linked to the underlying algebra, therefore a folding of the associated algebra –and the corresponding Dynkin diagrams –would be reflected to the structure of the Bethe equations. Extensive studies on the folding of the Bethe equations and the relevant Dynkin diagrams are presented in [21, 22, 31].

In a more physical frame this would be translated to a folding of the associated exact boundary S-matrices. Notwithstanding boundary S-matrices were extracted in [28] in the SP case, the derivation of boundary S-matrices in the SNP case is still an open question

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to date in the general case (see e.g. [17]). Having said this the derivation of the Bethe ansatz equations for trigonometric spin chains with SNP boundary conditions, and the associated boundary S-matrices will provide significant information at both physical and algebraic level.

The next natural step would be to identify the relevant boundary Lax pairs for both types of boundary conditions along the lines described in [42]. A comparison with the Lax pair constructed based on a set of postulates in [11] will be especially illuminating. In the SNP case the entailed Lax pair should coincide with that found in [11], whereas the Lax pair in the SP case will be of a novel from. Generalization of our results for any $A_n^{(1)}$ ($n > 1$) ATFT will be also presented in a separate publication. Finally, a similar exhaustive analysis regarding principal chiral models (partial results maybe found in [46]) will be particularly relevant especially bearing in mind the physical significance of a specific super-symmetric principal chiral model within the AdS/CFT correspondence [47, 48].

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A. Lie algebras

In this appendix we provide explicit expressions of the simple roots and the Cartan generators for $A_n^{(1)}$ [49]. The vectors $\alpha_i = (\alpha_i^1, \dots, \alpha_i^n)$ are the simple roots of the Lie algebra of rank n normalized to unity $\alpha_i \cdot \alpha_i = 1$, i.e.

$$\alpha_i = \left(0, \dots, 0, -\sqrt{\frac{i-1}{2i}}, \overset{i^{\text{th}}}{\underset{\downarrow}{\sqrt{\frac{i+1}{2i}}}}, 0, \dots, 0 \right), \qquad i \in \{1, \dots n\} \tag{A.1}$$

Also define the fundamental weights $\mu_k = (\mu_k^1, \dots, \mu_k^n), \quad k = 1, \dots, n$ as (see, e.g., [49]).

$$\alpha_j \cdot \mu_k = \frac{1}{2}\delta_{j,k}\,. \tag{A.2}$$

The extended (affine) root a_{n+1} is provided by the relation

$$\sum_{i=1}^{n+1} a_i = 0. \tag{A.3}$$

We give below the Cartan-Weyl generators in the defining representation:

$$\begin{aligned} E_{\alpha_i} &= e_{i\ i+1}\,, \qquad E_{-\alpha_i} = e_{i+1\ i}\,, \qquad E_{\alpha_n} = -e_{n+1\ 1}\,, \qquad E_{-\alpha_n} = -e_{1\ n+1} \\ H_i &= \sum_{j=1}^{n} \mu_j^i (e_{jj} - e_{j+1\ j+1})\,, \qquad\qquad i = 1, \dots, n \end{aligned} \tag{A.4}$$

For $A_2^{(1)}$ in particular we have:

$$\alpha_1 = (1,\ 0), \quad \alpha_2 = \left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right), \quad \alpha_3 = \left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right) \tag{A.5}$$

define also the following 3×3 generators

$$E_1 = E_{-1}^t = e_{12}, \quad E_2 = E_{-2}^t = e_{23}, \quad E_3 = E_{-3}^t = -e_{31} \tag{A.6}$$

where we define the matrices e_{ij} as $(e_{ij})_{kl} = \delta_{ik}\ \delta_{jl}$. The diagonal Cartan generators $H_{1,2}$ are

$$H_1 = \frac{1}{2}(e_{11} - e_{22}), \quad H_2 = \frac{1}{2\sqrt{3}}(e_{11} + e_{22} - 2e_{33}) \tag{A.7}$$

B. Evaluation of $W^{(k)}$, $\hat{W}^{(k)}$, $Z^{(k)}$, $\hat{Z}^{(k)}$

From the formulas (2.26), (2.29) the matrices $W^{(k)}$, $\hat{W}^{(k)}$, $Z^{(k)}$, $\hat{Z}^{(k)}$ may be determined. In particular, we write below explicit expressions of these matrices for the first orders.

$$W^{(0)} = \hat{W}^{(0)} = \begin{pmatrix} 0 & e^{\frac{i\pi}{3}} & 1 \\ e^{\frac{i\pi}{3}} & 0 & -1 \\ e^{\frac{2i\pi}{3}} & e^{-\frac{i\pi}{3}} & 0 \end{pmatrix},$$

$$\frac{m}{4}W^{(1)} = \begin{pmatrix} 0 & e^{\frac{2i\pi}{3}}a & c \\ -a & 0 & b \\ e^{\frac{i\pi}{3}}c & -b & 0 \end{pmatrix}, \qquad \frac{m}{4}\hat{W}^{(1)} = \begin{pmatrix} 0 & -\hat{b} & -\hat{a} \\ -e^{-\frac{i\pi}{3}}\hat{b} & 0 & -\hat{c} \\ \hat{a} & -e^{\frac{i\pi}{3}}\hat{c} & 0 \end{pmatrix}. \tag{B.1}$$

The higher order quantities are more complicated and we give the matrix entries below for $W^{(2)}$, $\hat{W}^{(2)}$ (define also, $\zeta = \frac{4}{m}$):

$$\begin{aligned}
W_{12}^{(2)} &= \frac{1}{3}(-2\gamma_3 + \gamma_1 + \gamma_2) + \frac{\zeta^2}{3}\left(2a' + b'\right) + \frac{\zeta^2}{3}\left(-2a^2 - bc\right), \\
W_{21}^{(2)} &= \frac{e^{-\frac{i\pi}{3}}}{3}(-2\gamma_3 + \gamma_1 + \gamma_2) + \frac{\zeta^2 e^{-\frac{i\pi}{3}}}{3}\left(a' - c'\right) + \frac{\zeta^2 e^{-\frac{i\pi}{3}}}{3}\left(c^2 - ab\right) \\
W_{13}^{(2)} &= \frac{1}{3}(-2\gamma_2 + \gamma_1 + \gamma_3) + \frac{\zeta^2}{3}\left(-b' + c'\right) + \frac{\zeta^2}{3}\left(b^2 - ac\right), \\
W_{31}^{(2)} &= \frac{1}{3}(2\gamma_2 - \gamma_1 - \gamma_3) + \frac{\zeta^2}{3}\left(-a' - 2c'\right) + \frac{\zeta^2}{3}\left(2c^2 + ab\right), \\
W_{23}^{(2)} &= -\frac{1}{3}(2\gamma_1 - \gamma_2 - \gamma_3) + \frac{\zeta^2}{3}\left(2b' + c'\right) + \frac{\zeta^2}{3}\left(-2b^2 - ac\right) \\
W_{32}^{(2)} &= -\frac{e^{\frac{i\pi}{3}}}{3}(2\gamma_1 - \gamma_2 - \gamma_3) + \frac{\zeta^2 e^{\frac{i\pi}{3}}}{3}\left(-a' + b'\right) + \frac{\zeta^2 e^{\frac{i\pi}{3}}}{3}\left(a^2 - bc\right)
\end{aligned} \tag{B.2}$$

and

$$\hat{W}_{12}^{(2)} = \frac{e^{-\frac{i\pi}{3}}}{3}(-2\gamma_2 + \gamma_1 + \gamma_3) + \frac{\zeta^2 e^{-\frac{i\pi}{3}}}{3}\left(\hat{b}' - \hat{c}'\right) + \frac{\zeta^2 e^{-\frac{i\pi}{3}}}{3}\left(\hat{c}^2 - \hat{a}\hat{b}\right),$$

$$
\begin{aligned}
\hat{W}_{21}^{(2)} &= \frac{1}{3}(-2\gamma_2+\gamma_1+\gamma_3)+\frac{\zeta^2}{3}\left(2\hat{b}'+\hat{a}'\right)+\frac{\zeta^2}{3}\left(-2\hat{b}^2-\hat{a}\hat{c}\right)\\
\hat{W}_{13}^{(2)} &= -\frac{1}{3}(-2\gamma_1+\gamma_3+\gamma_2)-\frac{\zeta^2}{3}\left(2\hat{a}'+\hat{c}'\right)+\frac{\zeta^2}{3}\left(2\hat{a}^2+\hat{b}\hat{c}\right),\\
W_{31}^{(2)} &= \frac{e^{\frac{i\pi}{3}}}{3}(2\gamma_1-\gamma_2-\gamma_3)+\frac{\zeta^2 e^{\frac{i\pi}{3}}}{3}\left(\hat{b}'-\hat{a}'\right)+\frac{\zeta^2 e^{\frac{i\pi}{3}}}{3}\left(-\hat{b}^2+\hat{a}\hat{c}\right),\\
\hat{W}_{23}^{(2)} &= -\frac{1}{3}(-2\gamma_3+\gamma_2+\gamma_1)+\frac{\zeta^2}{3}\left(\hat{a}'-\hat{c}'\right)+\frac{\zeta^2}{3}\left(-\hat{a}^2+\hat{b}\hat{c}\right)\\
\hat{W}_{32}^{(2)} &= \frac{1}{3}(-2\gamma_3+\gamma_1+\gamma_2)+\frac{\zeta^2}{3}\left(\hat{b}'+2\hat{c}'\right)+\frac{\zeta^2}{3}\left(-2\hat{c}^2-\hat{a}\hat{b}\right)
\end{aligned}
\tag{B.3}
$$

where the prime denotes derivative with respect to x, also $a,\ b,\ c$ are defined in (2.27) and have the following explicit form:

$$
\begin{aligned}
&a=\frac{\beta}{2}\left(\frac{\theta_1}{2}+\frac{\theta_2}{2\sqrt{3}}\right), \qquad b=\frac{\beta}{2}\left(-\frac{\theta_1}{2}+\frac{\theta_2}{2\sqrt{3}}\right), \qquad c=-\frac{\beta}{2}\frac{\theta_2}{\sqrt{3}},\\
&\gamma_1=e^{\beta\phi_1}, \qquad \gamma_2=e^{\beta(-\frac{1}{2}\phi_1+\frac{\sqrt{3}}{2}\phi_2)}, \qquad \gamma_3=e^{\beta(-\frac{1}{2}\phi_1-\frac{\sqrt{3}}{2}\phi_2)}.
\end{aligned}
\tag{B.4}
$$

Moreover using the expressions above and (2.29) we have:

$$
\begin{aligned}
\frac{dZ_{11}^{(1)}}{dx} &= \frac{e^{-\frac{i\pi}{3}}}{3}\frac{m}{4}(\gamma_1+\gamma_2+\gamma_3)+\frac{\zeta e^{-\frac{i\pi}{3}}}{3}\left(a'-c'\right)+\frac{\zeta e^{-\frac{i\pi}{3}}}{6}\left(a^2+b^2+c^2\right)\\
\frac{dZ_{22}^{(1)}}{dx} &= \frac{e^{\frac{i\pi}{3}}}{3}\frac{m}{4}(\gamma_1+\gamma_2+\gamma_3)+\frac{\zeta e^{\frac{i\pi}{3}}}{3}\left(b'-a'\right)+\frac{\zeta e^{\frac{i\pi}{3}}}{6}\left(a^2+b^2+c^2\right)\\
\frac{dZ_{33}^{(1)}}{dx} &= -\frac{1}{3}\frac{m}{4}(\gamma_1+\gamma_2+\gamma_3)-\frac{\zeta}{3}\left(c'-b'\right)-\frac{\zeta}{6}\left(a^2+b^2+c^2\right)\\
\frac{d\hat{Z}_{11}^{(1)}}{dx} &= \frac{e^{\frac{i\pi}{3}}}{3}\frac{m}{4}(\gamma_1+\gamma_2+\gamma_3)-\frac{\zeta e^{\frac{i\pi}{3}}}{3}\left(\hat{b}'-\hat{a}'\right)+\frac{\zeta e^{\frac{i\pi}{3}}}{6}\left(\hat{a}^2+\hat{b}^2+\hat{c}^2\right)\\
\frac{d\hat{Z}_{22}^{(1)}}{dx} &= \frac{e^{-\frac{i\pi}{3}}}{3}\frac{m}{4}(\gamma_1+\gamma_2+\gamma_3)+\frac{\zeta e^{-\frac{i\pi}{3}}}{3}\left(\hat{b}'-\hat{c}'\right)+\frac{\zeta e^{-\frac{i\pi}{3}}}{6}\left(\hat{a}^2+\hat{b}^2+\hat{c}^2\right)\\
\frac{d\hat{Z}_{33}^{(1)}}{dx} &= -\frac{1}{3}\frac{m}{4}(\gamma_1+\gamma_2+\gamma_3)+\frac{\zeta}{3}\left(\hat{a}'-\hat{c}'\right)-\frac{\zeta}{6}\left(\hat{a}^2+\hat{b}^2+\hat{c}^2\right)
\end{aligned}
\tag{B.5}
$$

Finally we report $Z_{ii}^{(2)},\ \hat{Z}_{ii}^{(2)}$:

$$
\begin{aligned}
\frac{dZ_{11}^{(2)}}{dx} &= \frac{e^{\frac{i\pi}{3}}}{3}\left(\gamma_2'-\gamma_3'-\zeta^2\left(c''-c^{2'}\right)+\zeta^2 ca'+(\gamma_1 c+\gamma_2 a+\gamma_3 b)-\zeta^2 abc\right)\\
\frac{dZ_{22}^{(2)}}{dx} &= \frac{e^{-\frac{i\pi}{3}}}{3}\left(-\gamma_1'+\gamma_3'-\zeta^2(a''-a^{2'})+\zeta^2 ab'+(\gamma_1 c+\gamma_2 a+\gamma_3 b)-\zeta^2 abc\right)\\
\frac{dZ_{33}^{(2)}}{dx} &= \frac{1}{3}\left(-\gamma_1'+\gamma_2'+\zeta^2(b''-b^{2'})-\zeta^2 bc'-(\gamma_1 c+\gamma_2 a+\gamma_3 b)+\zeta^2 abc\right)\\
\frac{d\hat{Z}_{11}^{(2)}}{dx} &= \frac{e^{-\frac{i\pi}{3}}}{3}\left(-\gamma_1'+\gamma_2'-\zeta^2(\hat{b}''-\hat{b}^{2'})+\zeta^2\hat{b}\hat{a}'+(\gamma_1\hat{c}+\gamma_2\hat{a}+\gamma_3\hat{b})-\zeta^2\hat{a}\hat{b}\hat{c}\right)\\
\frac{d\hat{Z}_{22}^{(2)}}{dx} &= \frac{e^{\frac{i\pi}{3}}}{3}\left(-\gamma_2'+\gamma_3'-\zeta^2(\hat{c}''-\hat{c}^{2'})+\zeta^2\hat{c}\hat{b}'+(\gamma_1\hat{c}+\gamma_2\hat{a}+\gamma_3\hat{b})-\zeta^2\hat{a}\hat{b}\hat{c}\right)\\
\frac{d\hat{Z}_{33}^{(2)}}{dx} &= \frac{1}{3}\left(-\gamma_1'+\gamma_3'+\zeta^2(\hat{a}''-\hat{a}^{2'})-\zeta^2\hat{a}\hat{c}'-(\gamma_1\hat{c}+\gamma_2\hat{a}+\gamma_3\hat{b})+\zeta^2\hat{a}\hat{b}\hat{c}\right).
\end{aligned}
\tag{B.6}
$$

C. Boundary terms

We present here the boundary contributions in the expansion of the classical open transfer matrix for both types of boundary conditions:

SNP boundary conditions. Recall that in this case the expansion of the generating function of the local integrals of motion is given in (3.3). After some tedious algebra we obtain for the boundary terms:

$$
\begin{aligned}
&\mathrm{f}_0 = \ln(3\bar{g}),\\
&\mathrm{f}_1 = \frac{1}{3\bar{g}}\Big(e^{\frac{\beta}{2}\alpha_3\cdot\Phi(0)}f_{31} - e^{\frac{\beta}{2}\alpha_2\cdot\Phi(0)}f_{23} - e^{\frac{\beta}{2}\alpha_1\cdot\Phi(0)}f_{12}\Big) + \frac{\zeta}{3}\Big(c(0) - b(0) + \hat{c}(0) - \hat{a}(0)\Big)\\
&\mathrm{f}_2 = -\frac{1}{3\bar{g}}\Big(f_{21}e^{-\frac{\beta}{2}\alpha_1\cdot\Phi(0)} + f_{32}e^{-\frac{\beta}{2}\alpha_2\cdot\Phi(0)} - f_{13}e^{-\frac{\beta}{2}\alpha_3\cdot\Phi(0)}\Big) \qquad \text{(C.1)}\\
&\quad -\frac{\zeta}{3\bar{g}}\Big(f_{12}e^{\frac{\beta}{2}\alpha_1\cdot\Phi(0)}(c(0)+\hat{c}(0)) - f_{23}e^{\frac{\beta}{2}\alpha_2\cdot\Phi(0)}b(0) + f_{31}e^{\frac{\beta}{2}\alpha_3\cdot\Phi(0)}\hat{a}(0)\Big)\\
&\quad -\frac{\zeta^2}{3}\Big(\hat{a}(0)c(0)+b(0)\hat{c}(0)\Big) + \frac{1}{3}\Big((2\gamma_1(0)-\gamma_2(0)-\gamma_3(0)) - \zeta^2(\hat{a}'(0)+b'(0)) + \zeta^2(\hat{a}^2(0)+b^2(0))\Big)\\
&\quad -\frac{1}{2}\Big(\frac{1}{3\bar{g}}\Big(e^{\frac{\beta}{2}\alpha_3\cdot\Phi(0)}f_{31} - e^{\frac{\beta}{2}\alpha_2\cdot\Phi(0)}f_{23} - e^{\frac{\beta}{2}\alpha_1\cdot\Phi(0)}f_{12}\Big) + \frac{\zeta}{3}(c(0) - b(0) + \hat{c}(0) - \hat{a}(0))\Big)^2.
\end{aligned}
$$

SP boundary conditions. We shall need for our purposes here the asymptotics of $K^{\pm}$ as $|u| \to \infty$:

$$
\begin{aligned}
K^+(|u| \to \infty,\ \xi^+) &\sim e_{33} - \frac{e^{-2i\xi^+}}{u}e_{11} + \frac{1}{u^2}e_{22} + \mathcal{O}(u^{-3})\\
K^-(|u| \to \infty,\ \xi^-) &\sim e_{11} - \frac{e^{-2i\xi^-}}{u}e_{33} + \frac{1}{u^2}e_{22} + \mathcal{O}(u^{-3}).
\end{aligned} \qquad \text{(C.2)}
$$

Then from the expansion of the boundary terms in (4.3), (4.4) we obtain the following explicit quantities:

$$
\begin{aligned}
&\mathrm{f}_0^+ = \mathrm{h}_0^+ = \ln[\frac{\Omega_{33}^2(0)}{3}], \quad \mathrm{f}_1^+ = -\zeta e^{\frac{i\pi}{3}}\hat{b}(0) + \zeta e^{-\frac{i\pi}{3}}c(0) - e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)},\\
&\mathrm{f}_2^+ = \Big\{\Omega_{22}^2(0)\Omega_{33}^{-2}(0) - \frac{e^{-4i\xi^+}}{2}\Omega_{11}^4(0)\Omega_{33}^{-4}(0) + \frac{\zeta e^{-2i\xi^+}}{2}\Omega_{11}^2(0)\Omega_{33}^{-2}(0)\,(c(0)+\hat{c}(0))\\
&\qquad -\frac{1}{6}\,(2\gamma_2(0)-\gamma_1(0)-\gamma_3(0)) - \frac{\zeta^2}{6}\big(b'(0)-c'(0)\big) + \zeta^2\Big(-\frac{c^2(0)}{6}+\frac{a^2(0)}{12}+\frac{b^2(0)}{12}+\frac{\hat{b}^2(0)}{4}\Big)\Big\}\\
&\qquad +i\sqrt{3}\Big\{\frac{\zeta e^{-2i\xi^+}}{2}\Omega_{11}^2(0)\Omega_{33}^{-2}(0)(\hat{c}(0)-c(0)) - \frac{1}{6}(2\gamma_2(0)-\gamma_1(0)-\gamma_3(0)) - \frac{\zeta^2}{6}\big(b'(0)-c'(0)\big)\\
&\qquad\qquad +\zeta^2\Big(-\frac{c^2(0)}{6}+\frac{a^2(0)}{12}+\frac{b^2(0)}{12}-\frac{\hat{b}^2(0)}{4}\Big)\Big\}\\
&\mathrm{h}_1^+ = -\zeta e^{\frac{i\pi}{3}}b(0) - e^{-2i\xi^+}e^{-\beta\alpha_3\cdot\Phi(0)}\\
&\mathrm{h}_2^+ = \Big\{\Omega_{22}^2(0)\Omega_{33}^{-2}(0) - \frac{e^{-4i\xi^+}}{2}\Omega_{11}^4(0)\Omega_{33}^{-4}(0) + \frac{\zeta e^{-2i\xi^+}}{2}\Omega_{11}^2(0)\Omega_{33}^{-2}(0)(c(0)+\hat{c}(0)) \qquad \text{(C.3)}
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{1}{3}(\gamma_1(0)-\gamma_2(0))+\frac{\zeta^2}{6}\left(\hat{b}'(0)-\hat{c}'(0)-b'(0)+a'(0)\right)\\
&+\zeta^2\left(\frac{\hat{a}^2(0)}{12}+\frac{\hat{b}^2(0)}{12}+\frac{\hat{c}^2(0)}{12}+\frac{b^2(0)}{6}-\frac{a^2(0)}{12}-\frac{c^2(0)}{12}\right)\Bigg\}\\
&+i\sqrt{3}\Bigg\{\frac{\zeta e^{-2i\xi^+}}{2}\Omega_{11}^2(0)\Omega_{33}^{-2}(0)\left(c(0)-\hat{c}(0)\right)+\frac{1}{6}\left(2\gamma_3(0)-\gamma_1(0)-\gamma_2(0)\right)\\
&\quad+\frac{\zeta^2}{6}\left(\hat{b}'(0)-\hat{c}'(0)+b'(0)-a'(0)\right)\\
&\quad+\zeta^2\left(\frac{\hat{a}^2(0)}{12}+\frac{\hat{b}^2(0)}{12}+\frac{\hat{c}^2(0)}{12}-\frac{b^2(0)}{6}+\frac{a^2(0)}{12}+\frac{c^2(0)}{12}\right)\Bigg\}
\end{aligned}
$$

Similar expressions are obtained for $\mathrm{f}_n^-,\ \mathrm{h}_n^-$:

$$
\begin{aligned}
\mathrm{f}_0^- &= \mathrm{h}_0^- = \ln\left[\frac{\Omega_{11}^{-1}(-L)}{3}\right], \qquad \mathrm{f}_1^- = \zeta e^{-\frac{i\pi}{3}}a(-L) - e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)}\\
\mathrm{f}_2^- =&\Bigg\{\Omega_{11}^2(-L)\Omega_{22}^{-2}(-L)-\frac{e^{-4i\xi^-}}{2}\Omega_{11}^4(-L)\Omega_{33}^{-4}(-L)\\
&\quad+\frac{\zeta e^{-2i\xi^-}}{2}\Omega_{11}^2(-L)\Omega_{33}^{-1}(-L)\left(a(-L)+\hat{a}(-L)\right)+\frac{1}{6}\left(2\gamma_2(-L)-\gamma_1(-L)-\gamma_3(-L)\right)\\
&\quad+\zeta^2\left(-\frac{a'(-L)}{18}-\frac{4c'(-L)}{18}+\frac{2b'(-L)}{18}\right)+\zeta^2\left(\frac{2c^2(-L)}{12}-\frac{a^2(-L)}{12}+\frac{2b^2(-L)}{12}\right)\Bigg\}\\
&+i\sqrt{3}\Bigg\{\frac{\zeta e^{-2i\xi^-}}{2}\Omega_{11}^2(-L)\Omega_{33}^{-2}(-L)(\hat{a}(-L)-a(-L))+\frac{1}{6}(2\gamma_2(-L)-\gamma_1(-L)-\gamma_3(-L))\\
&\quad+\zeta^2\left(-\frac{a'(-L)}{18}-\frac{4c'(-L)}{18}+\frac{2b'(-L)}{18}\right)+\zeta^2\left(\frac{2c^2(-L)}{12}-\frac{a^2(-L)}{12}+\frac{2b^2(-L)}{12}\right)\Bigg\}\\
\mathrm{h}_1^- &= -\zeta e^{-\frac{i\pi}{3}}\hat{b}(-L) - e^{-2i\xi^-}e^{-\beta\alpha_3\cdot\Phi(-L)}\\
\mathrm{h}_2^- =&\Bigg\{\Omega_{11}^2(-L)\Omega_{22}^{-2}(-L)-\frac{e^{-4i\xi^-}}{2}\Omega_{11}^4(-L)\Omega_{33}^{-4}(-L)\\
&\quad+\frac{\zeta e^{-2i\xi^-}}{2}\Omega_{11}^2(-L)\Omega_{33}^{-1}(-L)\left(a(-L)+\hat{a}(-L)\right)-\frac{1}{2}\left(\gamma_1(-L)-\gamma_2(-L)\right)\\
&\quad+\zeta^2\left(-\frac{\hat{b}'(-L)}{6}+\frac{\hat{c}'(-L)}{6}-\frac{a'(-L)}{6}+\frac{b'(-L)}{6}\right)\\
&\quad+\zeta^2\left(-\frac{\hat{c}^2(-L)}{12}-\frac{\hat{a}^2(-L)}{12}+\frac{\hat{b}^2(-L)}{6}+\frac{c^2(-L)}{12}+\frac{a^2(-L)}{12}+\frac{b^2(-L)}{12}\right)\Bigg\}\\
&+i\sqrt{3}\Bigg\{\frac{\zeta e^{-2i\xi^-}}{2}\Omega_{11}^2(-L)\Omega_{33}^{-2}(-L)(-\hat{a}(-L)+a(-L))+\frac{1}{6}(-2\gamma_3(-L)+\gamma_1(-L)+\gamma_2(-L))\\
&\quad+\zeta^2\left(-\frac{\hat{b}'(-L)}{6}+\frac{\hat{c}'(-L)}{6}+\frac{a'(-L)}{6}-\frac{b'(-L)}{6}\right)
\end{aligned}
$$

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$$+\zeta^2\left(-\frac{\hat{c}^2(-L)}{12}-\frac{\hat{a}^2(-L)}{12}+\frac{\hat{b}^2(-L)}{6}-\frac{a^2(-L)}{12}-\frac{b^2(-L)}{12}-\frac{c^2(-L)}{12}\right)\Bigg\}. \quad \text{(C.4)}$$

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